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A THESIS

UPON

THE FORMS OF QUARTICS WITH ONE NODE AND FOUR REAL AND DISTINCT ASYMPTOTES.

BY

EDWIN S. CRAWLEY.

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1. In Salmon's Higher Plane Curves (p. 213) quartics are classified as follows:

	m	δ	$\varkappa$	n	τ	ι	D
I	4	0	0	12	28	24	3
II	4	1	0	10	16	18	2
III	4	0	1	9	10	16	2
IV	4	2	0	8	8	12	1
V	4	1	1	7	4	10	1
VI	4	0	2	6	1	8	1
VII	4	3	0	6	4	6	0
VIII	4	2	1	5	2	4	0
IX	4	1	2	4	1	2	0
X	4	0	3	3	1	0	0

Where m represents the degree of the curve, δ the number of double points (crunodes and acnodes), $\varkappa$ the number of cusps, n the degree of the reciprocal curve, τ the number of double tangents, ι the number of inflexions and D the deficiency.

2. The number of different forms which a curve of the fourth degree may assume is of course very great, and has never been completely determined. The forms of non=singular quartics, that is those belonging to the first of the above groups have been studied by Zeuthen, (Math. Ann. VII 411) and the forms of those belonging to groups IV, V and VI have been studied under the name of "Bi=circular quartics" by various mathematicians. It is the object of the present paper to investigate the forms of certain curves belonging to group II.

3. If the nature of the infinite branches be made the basis of the classification the whole group may first be subdivided into three parts, as follows: The line at infinity will intersect the curve in four points: These may be

A - All real

B - Two real and two imaginary

C - All imaginary.

Each of these three classes is capable of subdivision. Thus in Class A the four real intersections of the line at infinity with the curve may be

1 - All distinct

2 - Two distinct and two coincident

3 - Two coincident and the other two coincident.

4 - Three coincident and one distinct

5 - Four coincident

The present paper has to do with only the first of these sub-divisions of Class A.

4. It will be convenient to take the node always at the intersection of the lines $x = 0$, $y = 0$ If then the ordinary rectangular system of coordinates be used the equations of all curves under consideration may be written

$$a_1 w_4 + a_2 w_3 + a_3 w_2 = 0 \qquad (1)$$

where w_n represents a homogeneous function of x and y of the n^{th} degree and a_n represents a real number. It is assumed that (1) has no linear nor quadratic factor. Hence (1) may be written in an expanded form thus:

$$\begin{aligned} a_1 (y - \alpha_1 x)(y - \alpha_2 x)(y - \alpha_3 x)(y - \alpha_4 x) \\ + a_2 (y - \beta_1 x)(y - \beta_2 x)(y - \beta_3 x) \\ + a_3 (y - \gamma_1 x)(y - \gamma_2 x) = 0 \end{aligned} \qquad (2)$$

where, to satisfy the conditions that have been imposed, the α's must be all real and distinct while the β's and γ's may have any values real or imaginary (except that the γ's must not be equal) consistent with the condition that w_3 and w_2 shall be real functions of x and y Of the numbers a_1, a_2 and a_3, a_2 is the only one that can vanish without destroying the nature of the curve.

5. Every line $y = \mu x$ through the origin intersects the curve (2) in two other points. These intersections may be real and different, real and coincident or imaginary. Hence the plane of the curve can be divided by lines radiating from the origin into regions in which alternately the curve is real and imaginary. The lines separating the real from the imaginary regions are tangents to the curve. It can easily be shown that the maximum number of such tangents is six; and hence the maximum number of the real and imaginary regions is three each. It is understood here that the portion of the plane on both sides of the origin lying between the same pair of tangents is one region. If in (2) $y = \mu x$ be substituted, we have

$$\begin{aligned} a_1(\mu-\alpha_1)(\mu-\alpha_2)(\mu-\alpha_3)(\mu-\alpha_4)x^2 \\ + a_2(\mu-\beta_1)(\mu-\beta_2)(\mu-\beta_3)x \\ + a_3(\mu-\gamma_1)(\mu-\gamma_2) = 0, \end{aligned} \tag{3}$$

which will be represented briefly by

$$p x^2 + q x + r = 0. \tag{4}$$

This equation will have equal roots if

$$q^2 - 4pr = 0 \tag{5}$$

Now (5) is a sextic in μ and hence cannot vanish for more than six values of μ. The values of μ which are roots of (5) determine lines, $y = \mu x$ which are tangent to the curve; hence the statement made above is proved. Since (5) may have only four, two or no real roots, it follows that the curve may lie in two real and two imaginary regions, in one real and one imaginary region, or the whole plane may form a real region. The curve can never be wholly imaginary because $4pr$ must vanish and change sign at least four times as μ passes from $+\infty$ to $-\infty$, and hence $q^2 - 4pr$ cannot be always negative.

6. The number of regions into which the plane of any given curve is divided can be quickly and easily determined by drawing the two sextic parabolas

$$q^2 = 0 \tag{6}$$

$$4pr = 0 \tag{7}$$

laying off the values of μ as abscissae and the values of the functions as ordinates. It then appears at a glance for what values of μ $q^2 - 4pr$ is < 0, $= 0$ or > 0. The figure ~~on the next page~~ below shows the general appearance of these sextics when the values of β and γ are all real and a_1 and a_3 have like signs. The curve AB represents $q^2 = 0$ and CD represents $4pr = 0$

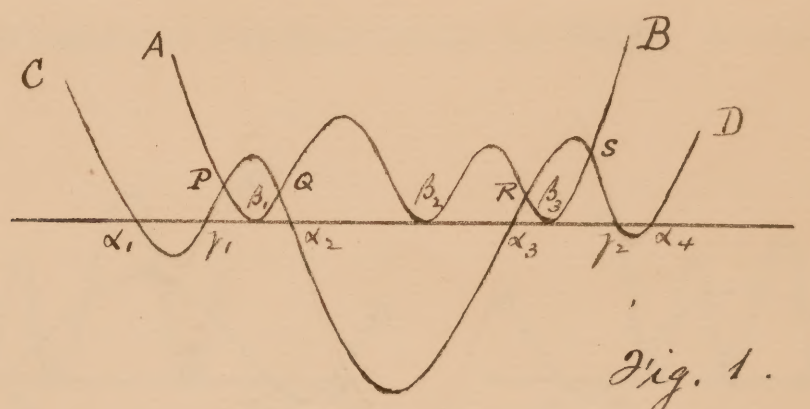

Fig. 1.

If a_1 and a_3 have unlike signs the curve CD extends to infinity below the horizontal axis instead of above. The modifications that occur when the values of γ are imaginary, and when any of the values of β are coincident, or when two of them are imaginary, are obvious. In the case of any given curve nothing more than a very rough construction of these sextics is required, their object being merely to show (1) into how many real and imaginary regions the plane of the curve is divided, and (2) what are the positions of the tangents dividing these regions with respect to the six values of μ indicated by α_1, α_2, α_3, α_4, γ_1, and γ_2.
In what follows these six values will be called the critical values of μ. It will be shown that the classification of the curves can be made to depend upon the arrangement of these six critical values in the real regions.

7. Before taking up the discussion of the problem stated in the last lines of the preceding paragraph it will be well to state some general deductions that can be drawn from a study of Fig. 1,

and to explain the notation that will be employed. The intersections of the curves, P, Q, R and S of course correspond to values of μ that divide the real from the imaginary regions. It is evident then that the number of critical values of μ lying in any real region must be always even. Again the number of intersections of the curves lying between any pair of critical values of μ must be always even. In case this number is four or six the sextics will have a form represented respectively by Figs. 2 and 3.

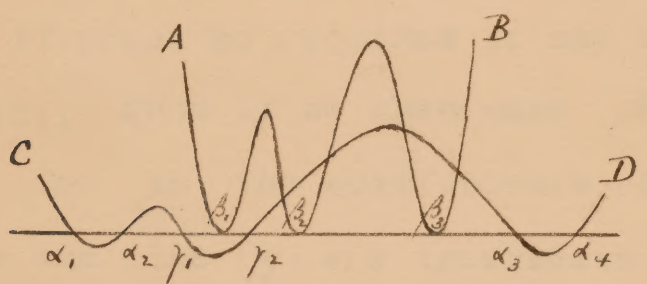

Fig. 2.

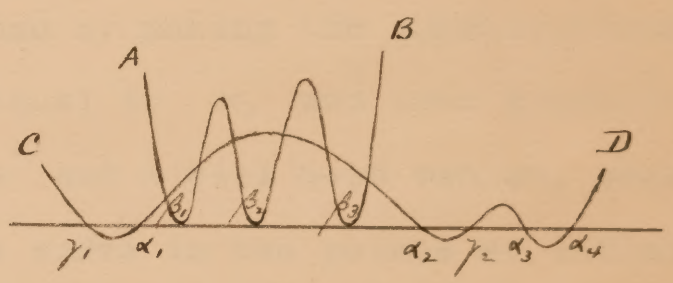

Fig. 3.

The facts that are shown by the two sextics in Fig. 1, can be represented just as well by the following symbol:

$$-\infty\,[\,\alpha_1\ \gamma_2\ t\ \text{imag.}\ t\ \alpha_2\ \alpha_3\ t\ \text{imag.}\ t\ \gamma_2\ \alpha_4\,]\,+\infty\,.$$

This will be understood to mean that when μ starts at $-\infty$ the line $y=\mu x$ intersects the quartic in real points; as μ increases and $y=\mu x$ revolves around, it becomes, when $\mu=\alpha_1$, parallel to the asymptote having this slope, and when $\mu=\gamma_1$, it becomes a tangent to one of the branches through the origin. After μ passes the value γ_1, $y=\mu x$ becomes a tangent to the curve and then passes into an imaginary region. Emerging from this imaginary region it becomes parallel in succession to the two asymptotes whose slopes are α_2 and α_3, after which it passes through tangency into another imaginary region. Coming out of this imaginary region it becomes tangent to the other branch through the origin when $\mu=\gamma_2$, parallel to the fourth asymptote when $\mu=\alpha_4$, and so on to $+\infty$ is the same real region in which it started. By the same notation the symbols corresponding to Figs. 2 and 3 will be respectively:

$$-\infty\,[\,\alpha_1\ \alpha_2\ \gamma_1\ \gamma_2\ t\ \text{imag.}\ t\ t\ \text{imag.}\ t\ \alpha_3\ \alpha_4\,]\,+\infty$$

$$-\infty\,[\,\gamma_1\ \alpha_1\ t\ \text{imag.}\ t\ t\ \text{imag.}\ t\ t\ \text{imag.}\ t\ \alpha_2\ \gamma_2\ \alpha_3\ \alpha_4\,]\,+\infty$$

It should be remarked that if ∞ occurs as one of the values of α or γ $4pr = o$ and its corresponding sextic will degrade to a quintic, but this in no way interferes with the deductions that have been drawn or with those that follow. If one or more of the values of α and β coincide the corresponding asymptotes will pass through the origin and they will then form dividing lines between real and imaginary regions. If a value of γ be equal to one of the values of β the branch tangent to $y = \gamma x$ will be inflected at the origin. These facts are sufficiently evident, but if proof be required it may be had by making the substitutions in (3). Thus if we there make β_1 equal to α, and then substitute α, for μ the coefficients of x^2 and x will both vanish, showing that the line $y = \alpha_1 x$ intersects the curve in two points at infinity and is therefore an asymptote. In the same manner if β_1 be made equal to γ_1 and we then substitute γ_1 for μ the roots of (3) are both O showing that all four intersections of $y = \gamma_1 x$ with the curve are at the origin. Since this line has simple intersection with one of the branches of the curve at the origin it must have three points in common with the other branch. That is it is tangent at a point of inflexion.

8. The different branches that, it will appear, can form parts of quartics belonging to the class under discussion will be designated as follows: An infinite branch lying along different asymptotes, whether it be inflected or not, will be called an hyperbola; such a curve with a loop, and therefore containing the double point at the origin, will be called a looped hyperbola; an infinite branch lying along one asymptote will be called a serpentine; an eight-shaped branch will be called a double loop; and any simple closed contour will be called an oval.

9. We are now prepared to investigate the relations between the forms of the curves and the arrangement of the critical values

of μ in the real regions. We shall consider first all the curves in which the values of γ are real, and afterwards those in which the values of γ are imaginary. When no critical value of μ falls in a given real region we have the symbol

$$imag.\ t_1\ t_2\ imag. \qquad (i)$$

From Figs. 2 and 3 it appears that throughout such a region $4pr$ is ppsitive. Hence the intersections of $y=\mu x$ with the quartic, which are given by

$$x = -\frac{q}{2p} \pm \frac{\sqrt{q^2-4pr}}{2p} \qquad (8)$$

will both have the same sign, that of $-\frac{q}{2p}$. Since in ~~the same~~ such a region neither p, q nor r can vanish, x is never 0 nor ∞. At t_1 and t_2 the values of x coincide and the line $y=\mu x$ is a tangent. The only form of curve that will fulfill these conditions is an <u>oval</u>, lying on one side of the origin. Moreover an oval cannot exist unless they are fulfilled, so that the condition is both necessary and sufficient.

(Note. In case μ passes through ∞ in the region considered the values of x will vanish simultaneously and change sign, but the further conclusion drawn above will still obtain; for by simply revolving the axes through a sufficient angle the values of μ in this region can be made all finite without shifting the origin with respect to the curve.)

10. When two critical values of μ fall in the same real region there are three cases to be investigated. These are represented by the symbols:

$$imag.\ t_1\ \alpha\ \gamma\ t_2\ imag. \qquad (ii)$$

$$imag.\ t_1\ \alpha_1\ \alpha_2\ t_2\ imag. \qquad (iii)$$

$$imag.\ t_1\ \gamma_1\ \gamma_2\ t_2\ imag. \qquad (iv)$$

In the real region represented by (ii) when $\mu=\alpha$, p vanishes and hence one value of x becomes infinite. When $\mu=\gamma$, r vanishes and one value of x is zero, the curve being tangent to $y=\gamma x$ at the

origin. Moreover the branch in this region must lie along the asymptote whose slope is α since it cannot pass off to any other asymptote without passing through an imaginary region. Hence the branch is a serpentine. It remains to show that these conditions are necessary to the existence of a serpentine as well as sufficient. A value of α must lie in any region containing a serpentine since the branch extends to infinity. A value of γ must lie in the same region since every serpentine must pass through the origin; for if we attempt to draw a serpentine that does not pass through the origin it will be found that the other asymptotes cannot be all used without intersecting the serpentine by at least one other branch of the curve and thus introducing a double point other than the one at the origin. To every serpentine through the origin it is possible to draw tangents whose slopes are related to α and γ as are the values t_1 and t_2 in *(ii)* Hence the conditions given are necessary.

The arrangement represented by *(iii)* gives a *pair* of opposite hyperbolas, because: (1), one value of x becomes ∞ when $\mu = \alpha_1$, and again when $\mu = \alpha_2$. (2), neither branch can pass off to another asymptote without passing through an imaginary region; (3), neither branch can pass through the origin because no value of γ lies in the region; (4), we have already seen that serpentines cannot exist without passing through the origin. It readily appears therefore that this condition is necessary for the existence of a *pair* of opposite hyperbolas as well as sufficient.

The arrangement represented by *(iv)* is the necessary and sufficient condition for the existence of a double loop. For the curve plainly cannot pass to infinity, since ρ cannot vanish, while it passes through the origin twice at the inclinations γ_1 and γ_2. No alteration is made in these deductions if in any of these three cases α or γ coincides with the adjacent values of t.

11. The next cases to take up are those in which four critical

values of μ lie in the same real region. We can write the following seven essentially different arrangements.

$$\text{imag. } t_1\ \alpha_1\ \alpha_2\ \alpha_3\ \alpha_4\ t_2\ \text{imag.} \qquad (v)$$

$$\left.\begin{array}{l}\text{imag. } t_1\ \gamma_1\ \alpha_1\ \alpha_2\ \alpha_3\ t_2\ \text{imag.}\\ \text{imag. } t_1\ \alpha_1\ \gamma_1\ \alpha_2\ \alpha_3\ t_2\ \text{imag.}\end{array}\right\} \qquad (vi)$$

$$\left.\begin{array}{l}\text{imag. } t_1\ \gamma_1\ \gamma_2\ \alpha_1\ \alpha_2\ t_2\ \text{imag.}\\ \text{imag. } t_1\ \gamma_1\ \alpha_1\ \gamma_2\ \alpha_2\ t_2\ \text{imag.}\\ \text{imag. } t_1\ \gamma_1\ \alpha_1\ \alpha_2\ \gamma_2\ t_2\ \text{imag.}\\ \text{imag. } t_1\ \alpha_1\ \gamma_1\ \gamma_2\ \alpha_2\ t_2\ \text{imag.}\end{array}\right\} \qquad (vii)$$

The second and third arrangements, however, give the same form of curve, and the same is true of the last four. In view of the fact that *(v)* must occur in any given curve in connection with *(iv)* it must give only infinite branches that do not pass through the origin. Hence it will always give <u>four hyperbolas</u>, which of course do not intersect and which are <u>not</u> arranged in opposite pairs.

The arrangement *(vi)* always occurs in combination with *(ii)* Hence one of the branches represent by *(vi)* must pass through the origin, and, since it cannot be a serpentine it must be an hyperbola The other two branches must of course be hyperbolas. Hence *(vi)* gives always <u>three hyperbolas</u>, one of which passes through the origin. Since both values of γ lie in the real region represented by *(vii)* the double point at the origin must lie upon the branches given by this combination. In view of what has gone before these branches must be either two intersecting hyperbolas or a looped hyperbola with a simple hyperbola. But two intersecting hyperbolas are impossible for if two such curves confined to the same pair of asymptotes, as these are, intersect once they must intersect again and we should have two double points. Hence *(vii)* gives always one <u>simple hyperbola</u> and a <u>looped hyperbola</u>.

12. When all six critical values of μ lie in the same real region there will be a number of possible arrangements of the α's and γ's but it is needless to specify them all. An example of such an arrangement is

$$imag.\ t_1\ \alpha_1\ \gamma_1\ \alpha_2\ \gamma_2\ \alpha_3\ \alpha_4\ t_2\ imag. \qquad (viii)$$

From the absence of the conditions necessary for all the other forms the only two that are possible here are either four simple hyperbolas or three simple hyperbolas and a looped hyperbola. A further remark upon this case will be made in §14

13. When all the intersections of the line $y=\mu x$ with the curve are real the symbol will be

$$-\infty\ [\alpha_1\ \alpha_2\ \gamma_1\ \alpha_3\ \gamma_2\ \alpha_4\ ]+\infty \qquad (ix)$$

or a similar arrangement with a different permutation of the α's and γ's. In this case too all that can be said at present is that it will give either four simple hyperbolas, or three simple hyperbolas and a looped hyperbola.

14. It would now be proper to investigate the conditions under which *(viii)* and *(ix)* will give looped hyperbolas. Want of time has prevented me from doing this, except in part. Enough of the investigation is given below in this section to answer the immediate purpose of this paper, that is to determine the possible forms of curves belonging to the class in question. Should opportunity be given to complete the theory it will be added in the form of an appendix.

In this paragraph when it is stated that μ changes from γ_1 to γ_2 it is always understood that during the whole change the intersections of $y=\mu x$ with the curve are real.

1st. Let no value of α or β lie between γ_1 and γ_2, then of the two quantities:

$$x_1 = -\frac{q}{2p} + \frac{\sqrt{q^2-4pr}}{2p},$$

and

$$x_2 = -\frac{q}{2p} - \frac{\sqrt{q^2-4pr}}{2p}.$$

one ($say\ x_1$) is zero when $\omega = \gamma_1$, (for then $n = 0$) and when $\omega = \gamma_2$ (for the same reason), and is finite and continuous between these limits. Hence it must describe a loop.

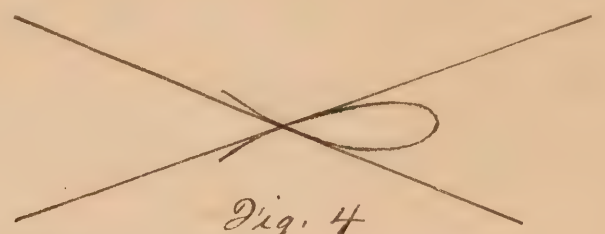

Fig. 4

2nd. Let one or more values of α lie between γ_1 and γ_2. In this case ϕ will change sign at each value of α as ω changes from γ_1 to γ_2, but it is plain that one of the values of x ($say\ x_1$) will still fulfill the conditions stated above, and hence a loop will be formed.

3rd. Let a value of β lie between γ_1 and γ_2 then when $\omega = \gamma_1$, one value of x (say x_1) will vanish while for $\omega = \gamma_2$, since y has changed sign, x_2 will be zero and x_1 will be finite. The figure is shown below:

Fig. 5

Now if, after passing the value γ_2, ω comes next to a value of α lying in the same real region x, will continue increasing and go off to infinity and there will be no loop; but if, after passing γ_2, ω approaches a value that makes $y = \omega x$ a tangent the two values x_1 and x_2 will coalesce and form a loop, as in Fig. 6.

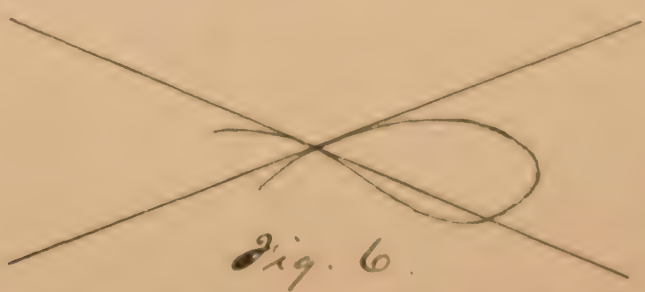

Fig. 6

Hence we may conclude that if in any real region that contains both values of γ no values of β is found then a loop will be formed, or if one value of β (but no value of α) has such a position, but one or both values of x are adjacent to imaginary

regions the same form will result. If, however, one value of β lies between the $\gamma's$ and the latter are neither adjacent to imaginary regions there will be no loop. If two values of β lie between the $\gamma's$ the result is the same as when no value of β has such a position, for by the time ω reaches the value γ_2, y has again the same sign that it had when $\omega = \gamma_1$. In the same way three values of β have the same effect as one.

15. An enumeration can now be made of the possible forms of curves in which the node is the intersection of real branches. This is done by making every possible consistent combination of the symbols given in §§ 9 – 13. The result is given below in a table. The following abbreviations are used.

h. = hyperbola s. = serpentine

$o.h.$ = opposite hyperbolas $d.l.$ = double loop

$l.h.$ = looped hyperbola $o.$ = oval.

Table I

	$h.$	$o.h.$ (pairs)	$l.h.$	$s.$	$d.l.$	$o.$	Symbols
a		1		2			(ii), (ii') and (ii'')
b		2			1		(iii), (iii') and (iv)
c	4				1		(iv) and (v)
d	4				1	1	(iv), (v) and (i)
e	3			1			(ii) and (vi)
f	3			1		1	(ii'), (vi) and (i)
g	1	1	1				(iii') and (vii)
h	1	1	1			1	(iii'), (vii') and (i)
i	4						(viii') or (ix)
k	4					1	(viii') and (i)
l	3		1				(viii') or (ix)
m	3		1			1	(viii') and (i)
n	3		1			2	(viii'), (i) and (i)

It will be noticed that the combination *(viii'), (i') and (i')* giving 4 hyperbolas and 2 ovals is omitted. But it must be remembered that for *(viii')* to give four hyperbolas a value of β must lie in a real region between the two values of γ. But when two ovals exist all three values of β must lie between two critical values of μ dividing that space into three imaginary and two real regions. Hence in this combination *(viii')* must always give a looped hyperbola.

Examples of each of these forms are given in the accompanying diagrams. See § 19.

16. So far no account has been taken of the condition $a_2 = 0$ If this condition is satisfied equation (3) becomes:

$$a_1(\mu-\alpha_1)(\mu-\alpha_2)(\mu-\alpha_3)(\mu-\alpha_4)\chi^2 + a_3(\mu-\gamma_1)(\mu-\gamma_2) = 0 \qquad (9)$$

or

$$\chi = \pm\sqrt{\frac{-a_3(\mu-\gamma_1)(\mu-\gamma_2)}{a_1(\mu-\alpha_1)(\mu-\alpha_2)(\mu-\alpha_3)(\mu-\alpha_4)}} \qquad (10)$$

From (10) it appears that the quantity under the radical changes sign every time μ passes through a critical value; hence the asymptotes (all of which pass through the node) and the tangents at the node are the lines that divide the plane of the curve into real and imaginary regions.

The only different symbols possible are the following

$$-\infty \left[\text{imag.}\ \overset{t}{\alpha_1}\ \overset{t}{\alpha_2}\ \text{imag}\ \overset{t}{\gamma_1}\ \overset{t}{\gamma_2}\ \text{imag.}\ \overset{t}{\alpha_3}\ \overset{t}{\alpha_4}\ \text{imag}\right] +\infty$$

$$-\infty \left[\text{imag.}\ \overset{t}{\alpha_1}\ \overset{t}{\gamma_1}\ \text{imag.}\ \overset{t}{\alpha_2}\ \overset{t}{\alpha_3}\ \text{imag.}\ \overset{t}{\gamma_2}\ \overset{t}{\alpha_4}\ \text{imag}\right] +\infty$$

which, by the criteria already established (§ 10) give in the first case 2 pairs of opposite hyperbolas and a double loop, and in the second case one pair of opposite hyperbolas and 2 serpentines.

The curves therefore belong respectively to forms 7 and 2 Table I. They will be under the present conditions symmetrical about the origin.

17. Consideration of the case where γ_1 and γ_2 are imaginary. The equation of the curve may be written in the form:

$$a_1(y-\alpha_1 x)(y-\alpha_2 x)(y-\alpha_3 x)(y-\alpha_4 x) + a_2(y-\beta_1 x)(y-\beta_2 x)(y-\beta_3 x) + a_3(y^2+sxy+tx^2)=0 \qquad (11)$$

where $y^2+sxy+tx^2$ has no real linear factor.

The equation in x and μ is

$$a_1(\mu-\alpha_1)(\mu-\alpha_2)(\mu-\alpha_3)(\mu-\alpha_4)x^2 + a_2(\mu-\beta_1)(\mu-\beta_2)(\mu-\beta_3)x + a_3(\mu^2+s\mu+t)=0 \qquad (12)$$

As before writing (12) in the form

$$p x^2+qx+r=0$$

the values of x will be equal if

$$q^2-4pr=0$$

and the conditions governing the separation of the plane into real and imaginary regions are unaltered under the present hypothesis. The sextic parabola

$$4pr=0$$

can have now, however, but four intersections with the horizontal axis and the figures will be somewhat as follows:

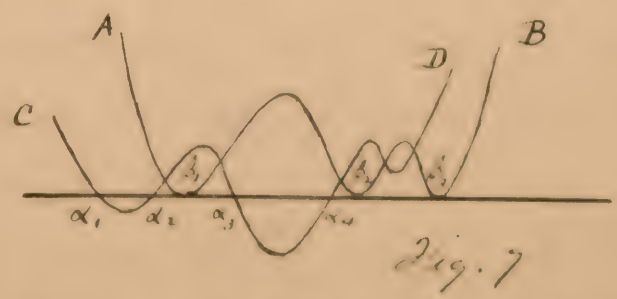

Fig. 7

AB is the curve of $q^2=0$ and CD of $4pr=0$ Under the present conditions the curves must consist exclusively of hyperbolas and ovals, with of course the acnode at the origin. The symbols of the different curves are therefore:

$o.$ $\quad -\infty\,[\,imag.\ t\ \alpha_1\ \alpha_2\ t\ imag.\ t\ \alpha_3\ \alpha_4\ t\ imag\,]+\infty$

$p.$ $\quad -\infty\,[\,imag.\ t\ \alpha_1\ \alpha_2\ t\ imag.\ t\ \alpha_3\ \alpha_4\ t\ imag.\ t\ t\ imag\,]+\infty$

$q.$ $\quad -\infty\,[\,imag.\ t\ \alpha_1\ \alpha_2\ \alpha_3\ \alpha_4\ t\ imag.\,]+\infty$

$r.$ $\quad -\infty\,[\,imag.\ t\ \alpha_1\ \alpha_2\ \alpha_3\ \alpha_4\ t\ imag\ t\ t\ imag.\,]+\infty$

$s.$ $\quad -\infty\,[\,imag.\ t\ \alpha_1\ \alpha_2\ \alpha_3\ \alpha_4\ t\ imag.\ t\ t\ imag\ t\ t\ imag.\,]+\infty$

Hence we have:

$o.$ 2 pairs of opp. hyperbolas and acnode

$p.$ 2 " " " " 1 oval and acnode

$q.$ 4 hyperbolas and acnode

$r.$ 4 " 1 oval and acnode

$s.$ 4 " 2 ovals and acnode.

Examples of these forms are also given.

18. If $a_2 = o$, χ is determined by

$$\chi = \pm\sqrt{\frac{-a_3(\mu^2+s\mu+t)}{a_1(\mu-\alpha_1)(\mu-\alpha_2)(\mu-\alpha_3)(\mu-\alpha_4)}}$$

From this it appears that the plane is divided into four regions, two real and two imaginary. The symbol for this case is

$$imag.\ \overset{t}{\alpha_1}\ \overset{t}{\alpha_2}\ imag.\ \overset{t}{\alpha_3}\ \overset{t}{\alpha_4}\ imag.$$

and the curve therefore consists of two pairs of opposite hyperbolas and the acnode. It therefore belongs under species (o).

19. It thus appears that there are eighteen different forms of quartics having four real and distinct asymptotes and one double point real or imaginary. The following eighteen equations give curves, illustrating the different forms.

$a.$ $\quad 2y^4+3xy^3-3x^2y^2-2x^3y+2y^3+5xy^2+2x^2y+5y^2$
$\qquad -10xy-3x^2-0$

$b.$ $\quad 2xy^3-3x^2y^2-5x^3y+6x^4+2y^3-3xy^2-3x^2y+2x^3$
$\qquad +y^2+xy=0$

c. $2y^4+9xy^3+3x^2y^2-4x^3y+y^3+5xy^2+2x^2y-8x^3$
$-3xy+3x^2=0$

d. $\frac{1}{8}(2y^3x-3y^2x^2-5yx^3+6x^4)+3y^3-3yx^2+y^2+xy=0$

e. $2y^4+3xy^3-3x^2y^2-2x^3y+2y^3-3xy^2-3x^2y+2x^3$
$+8y^2-10xy-3x^2=0$

f. $y^3x-y^2x^2-2yx^3+5y^3-5yx^2-y^2-yx+\frac{1}{2}x^2=0$

g. $4y^4+12xy^3-7x^2y^2-18x^3y+9x^4+y^3+x^2y^2-4x^2y-4x^3$
$+4y^2+3xy=0$

h. $xy^3-x^3y+4y^3-4x^2y+y^2-\frac{25}{16}x^2=0$

i. $y^3x+y^2x^2-2yx^3+3y^3-9yx^2-6x^3+2y^2-7xy-4x^2=0$

k $y^3x+y^2x^2-2yx^3+2y^3+8y^2x-10yx^2+y^2-xy-2x^2=0$

l $y^4+3y^3x-y^2x^2-3yx^3+4y^3-12y^2x-16yx^2+48x^3$
$+2y^2-19yx-10x^2=0$

m $y^3x-3y^2x^2+2yx^3+3y^3+3y^2x-6yx^2+y^2+2yx+\frac{3}{4}x^2=0$

n. $xy^3+2x^2y^2-x^3y-2x^4+5y^3+\frac{5}{4}xy^2-5x^2y-\frac{5}{4}x^3$
$+\frac{1}{2}y^2-\frac{25}{32}x^2=0$

o. $y^3x-yx^3+y^3-yx^2+y^2+xy+x^2=0$

p $xy^3-x^3y+4y^3-4x^2y+y^2+xy+x^2=0$

q $y^3x-yx^3+y^3+4y^2x+4yx^2+3x^3+y^2-xy+3x^2=0$

r $y^3x-y^2x^2-2yx^3+5y^3-5yx^2+y^2+xy+x^2=0$

s. $y^3x+2y^2x^2-yx^3-2x^4+5y^3+\frac{5}{4}y^2x-5yx^2-\frac{5}{4}x^3$
$-\frac{1}{2}y^2-\frac{1}{4}x^2=0$

20. In addition to the conditions governing the existence of looped hyperbolas, which have been only partly investigated, the question of the effect upon the forms of these curves caused by special forms of contact between the curve and the asymptotes

at infinity--contact at points of infexion or undulation--is reserved for future consdieration.

University of Pennsylvania
May 1892.

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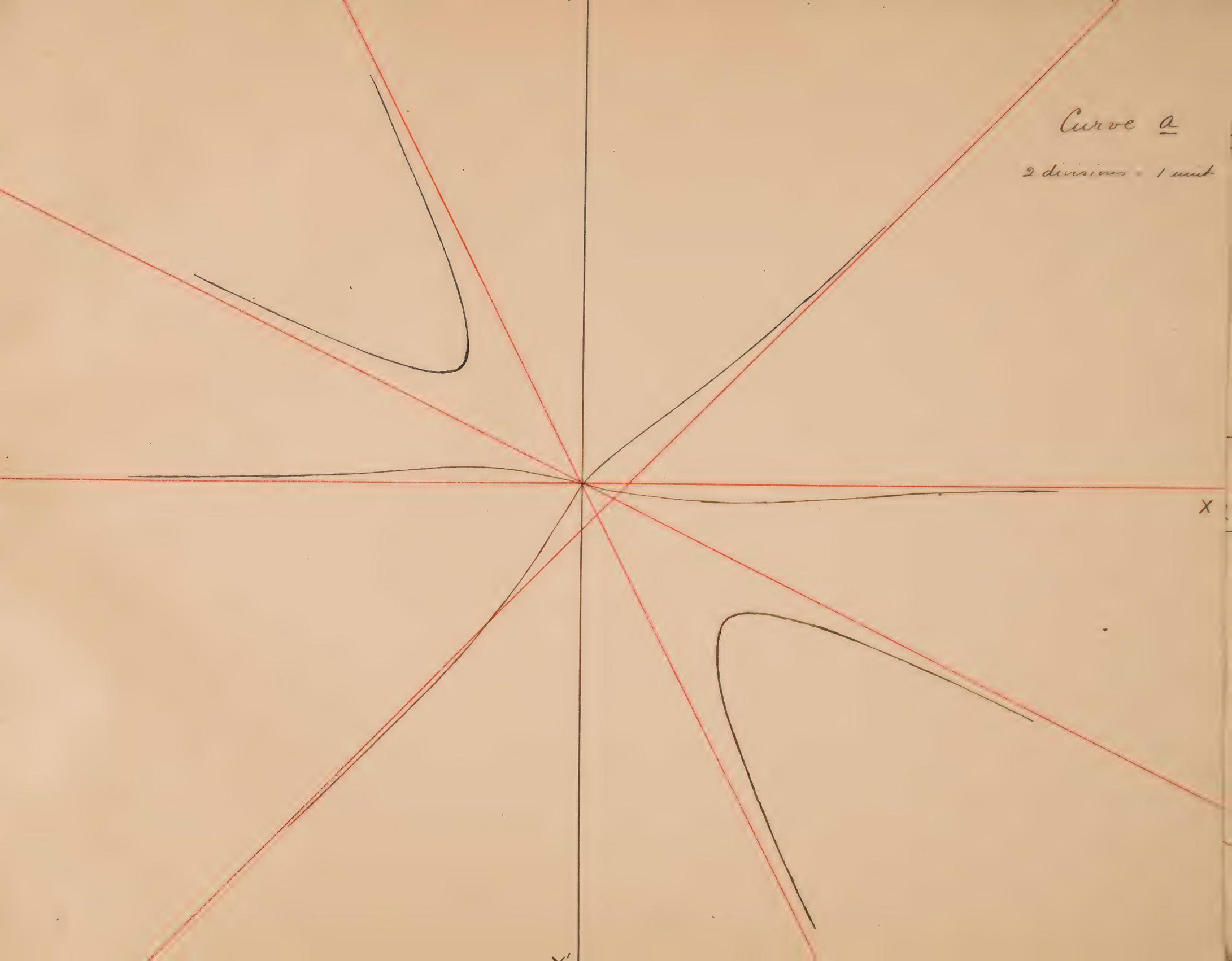
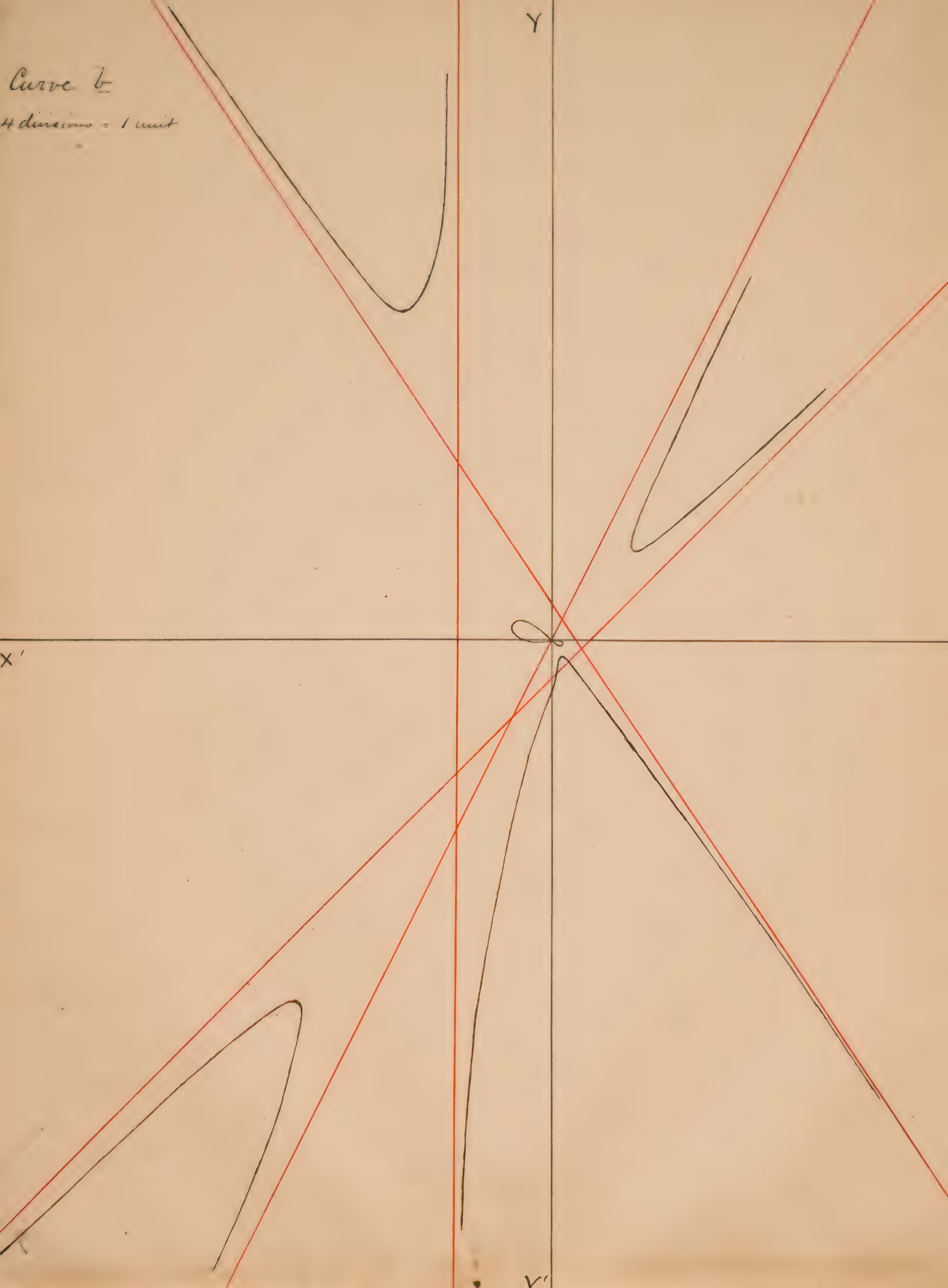
Diagrams to accompany Thesis upon the Forms of Quartics, by

Edwin S. Crawley.


Curve a
2 divisions = 1 unit
X
X'

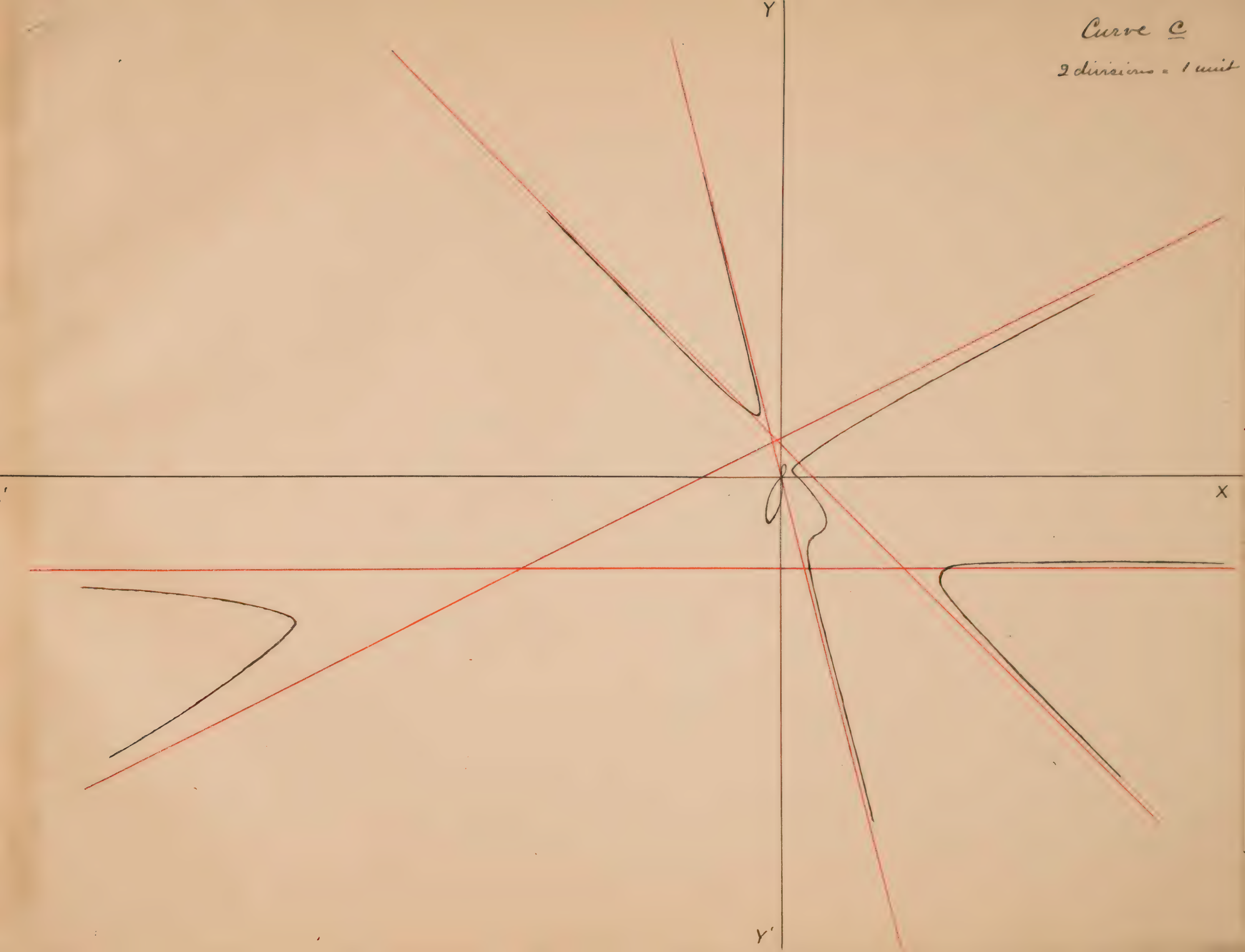
Curve b

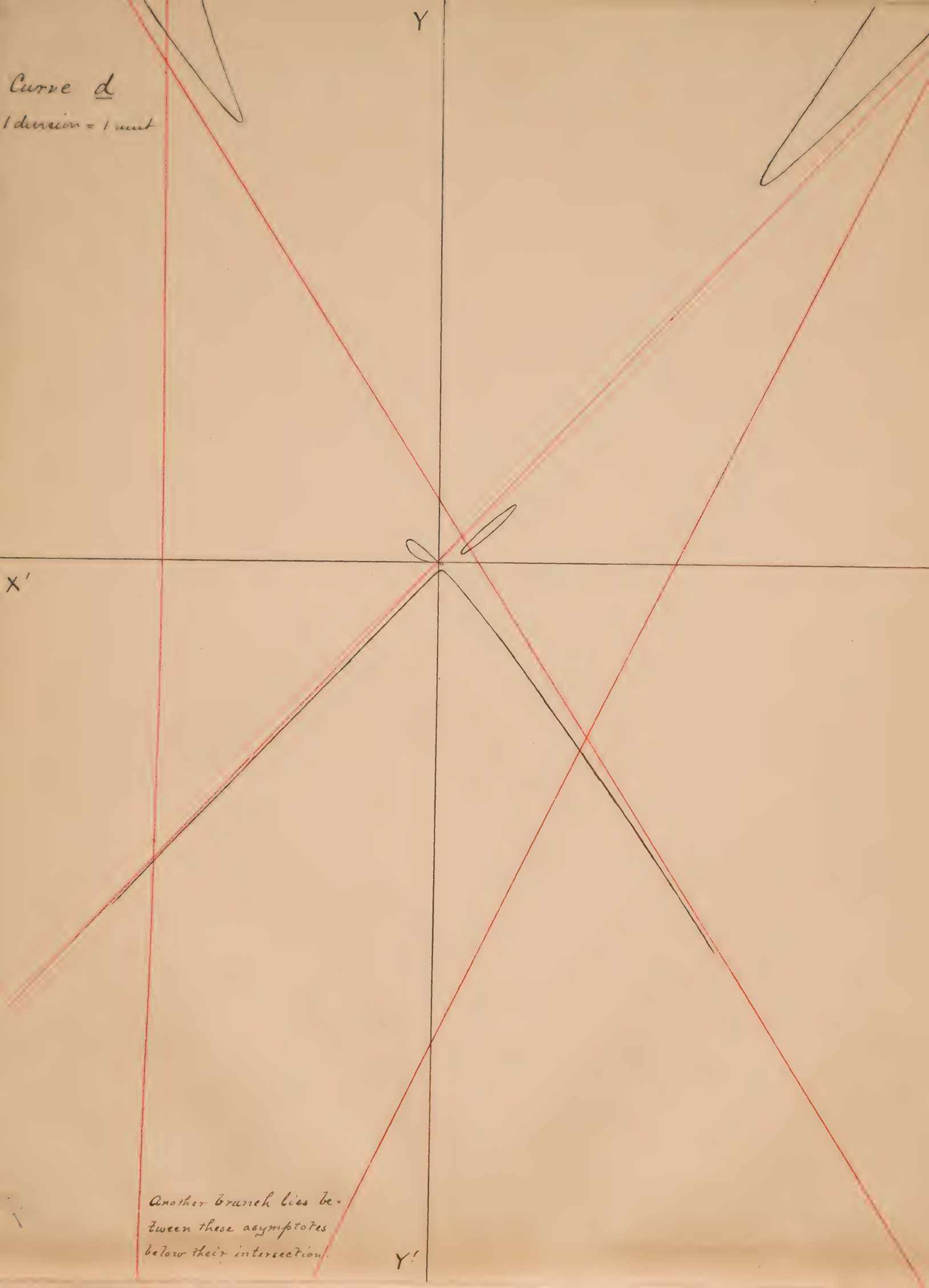
4 divisions = 1 unit



Courbe C

2 divisions = 1 unité




Curve d
1 division = 1 unit
Y
X'
Another branch lies be-
tween these asymptotes
below their intersection.
Y'

Y

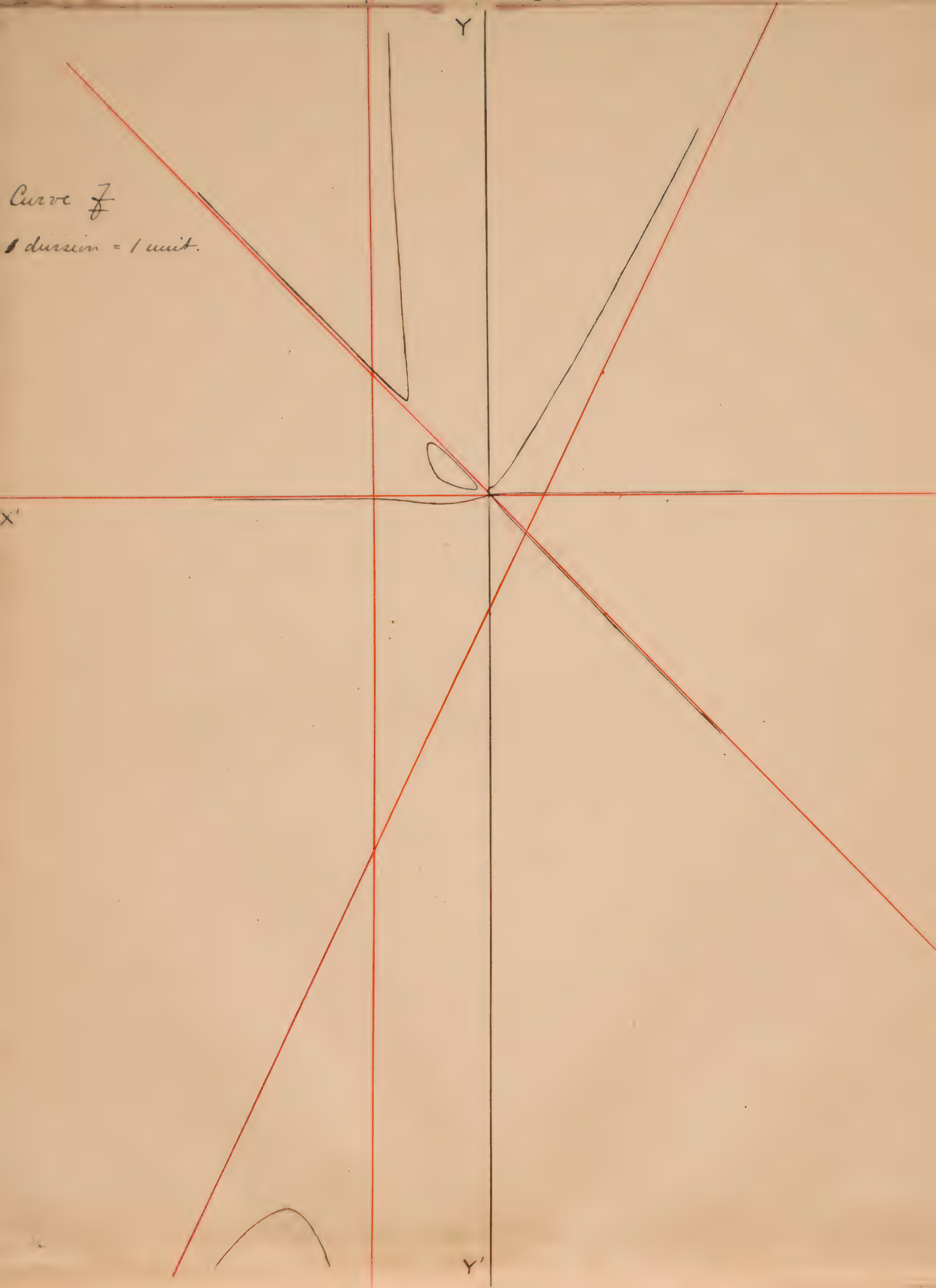
Curve e

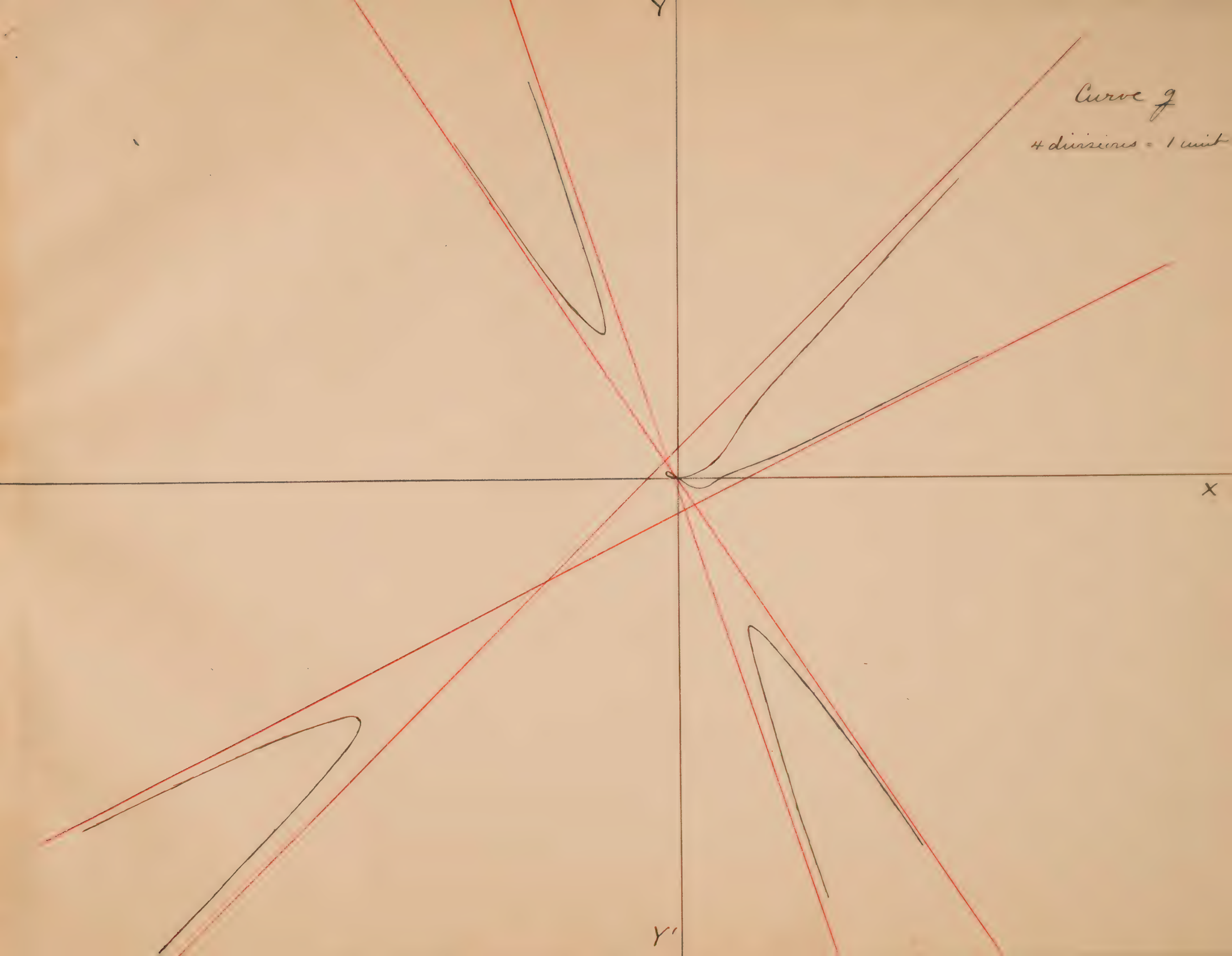
2 divisions - 1 unit

X'

X

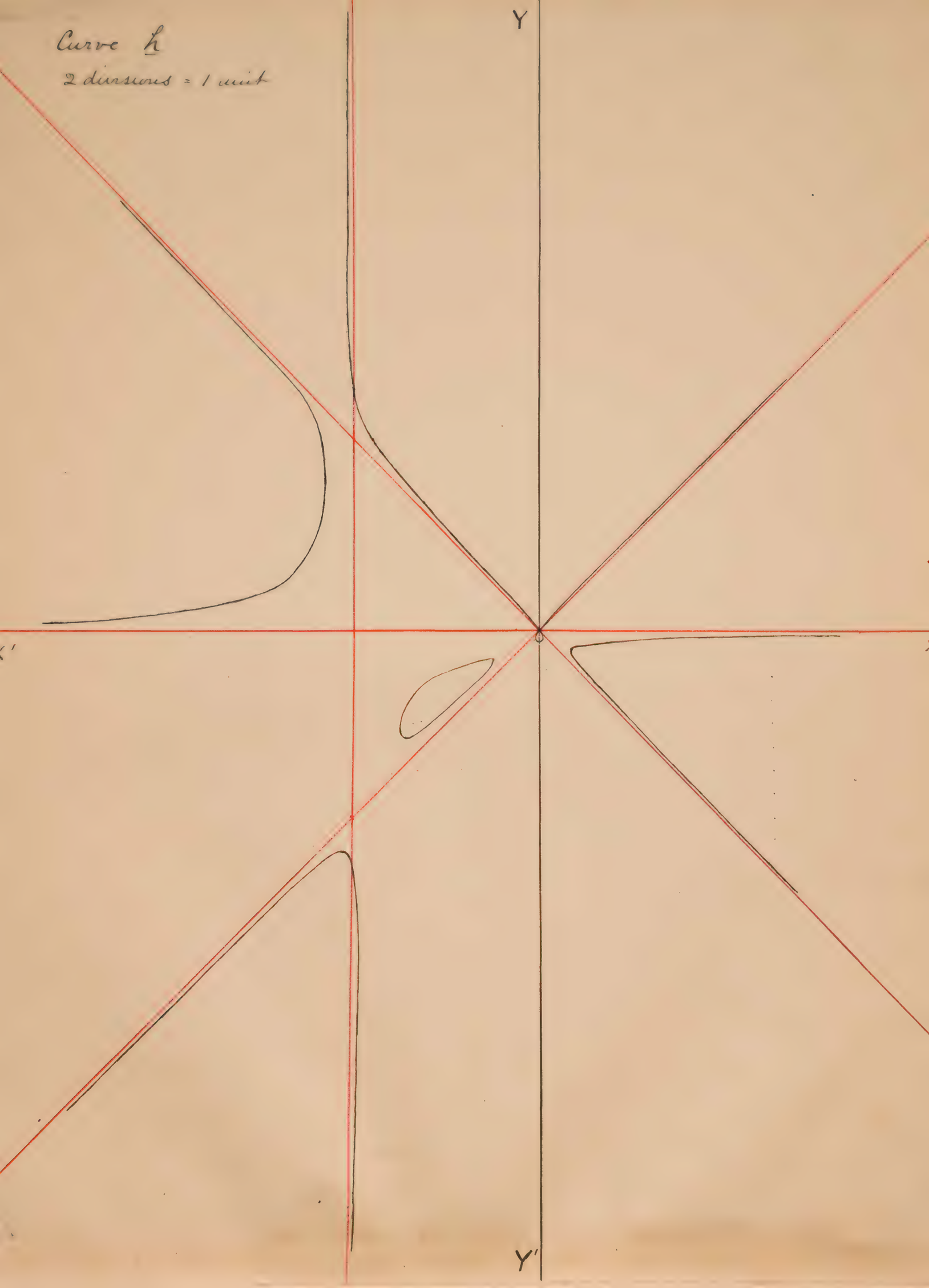
Y'


Curve 7
1 division = 1 unit.
Y
X'
Y'


Y
X
Y'
Curve 7
4 divisions = 1 unit

Curve k

2 divisions = 1 unit



Y

Courbe i

1 division = 1 unité

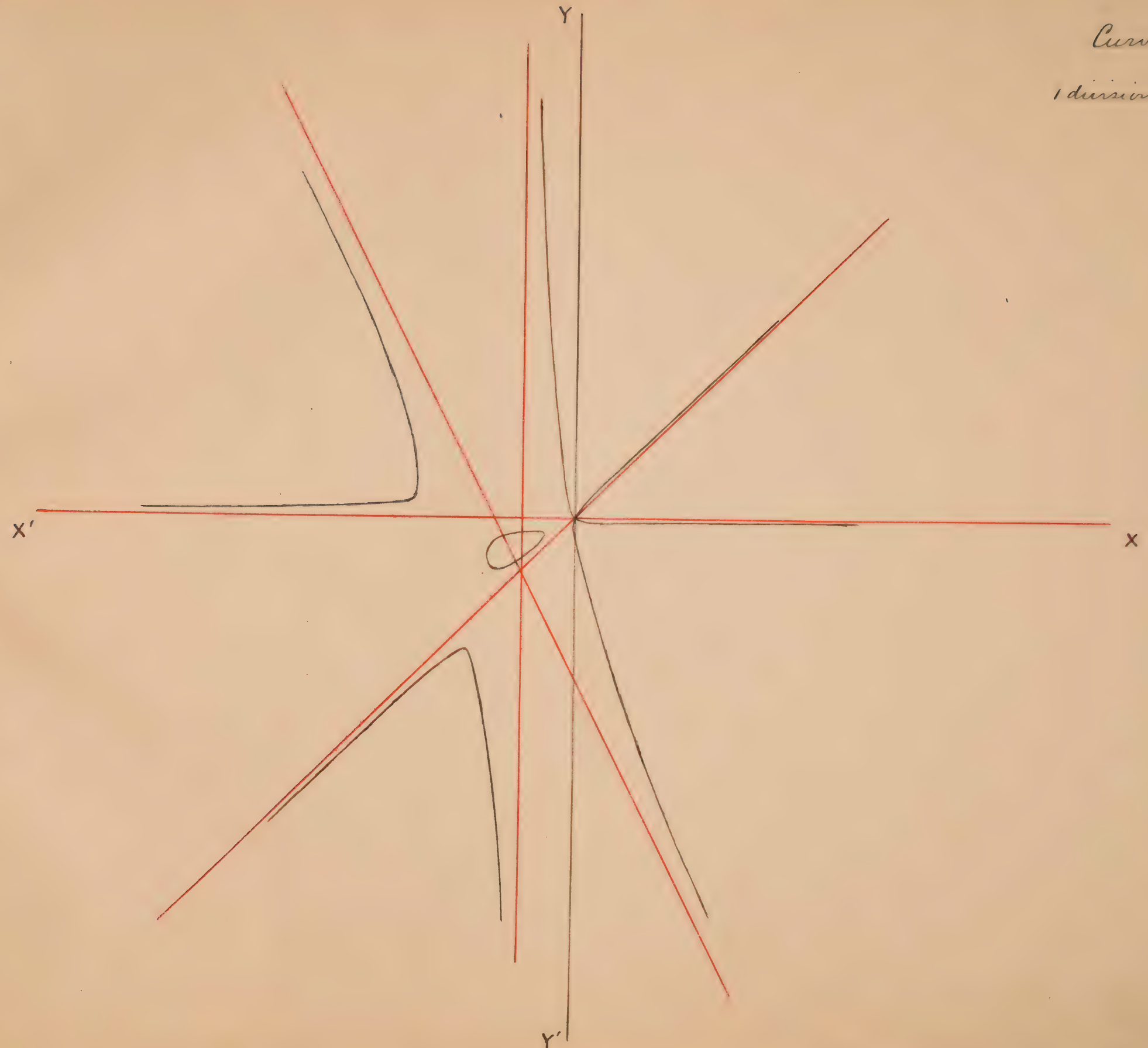
X'

X

Y'

Curve k

1 division = 1 unit



Y

Courbe $\underline{l}$

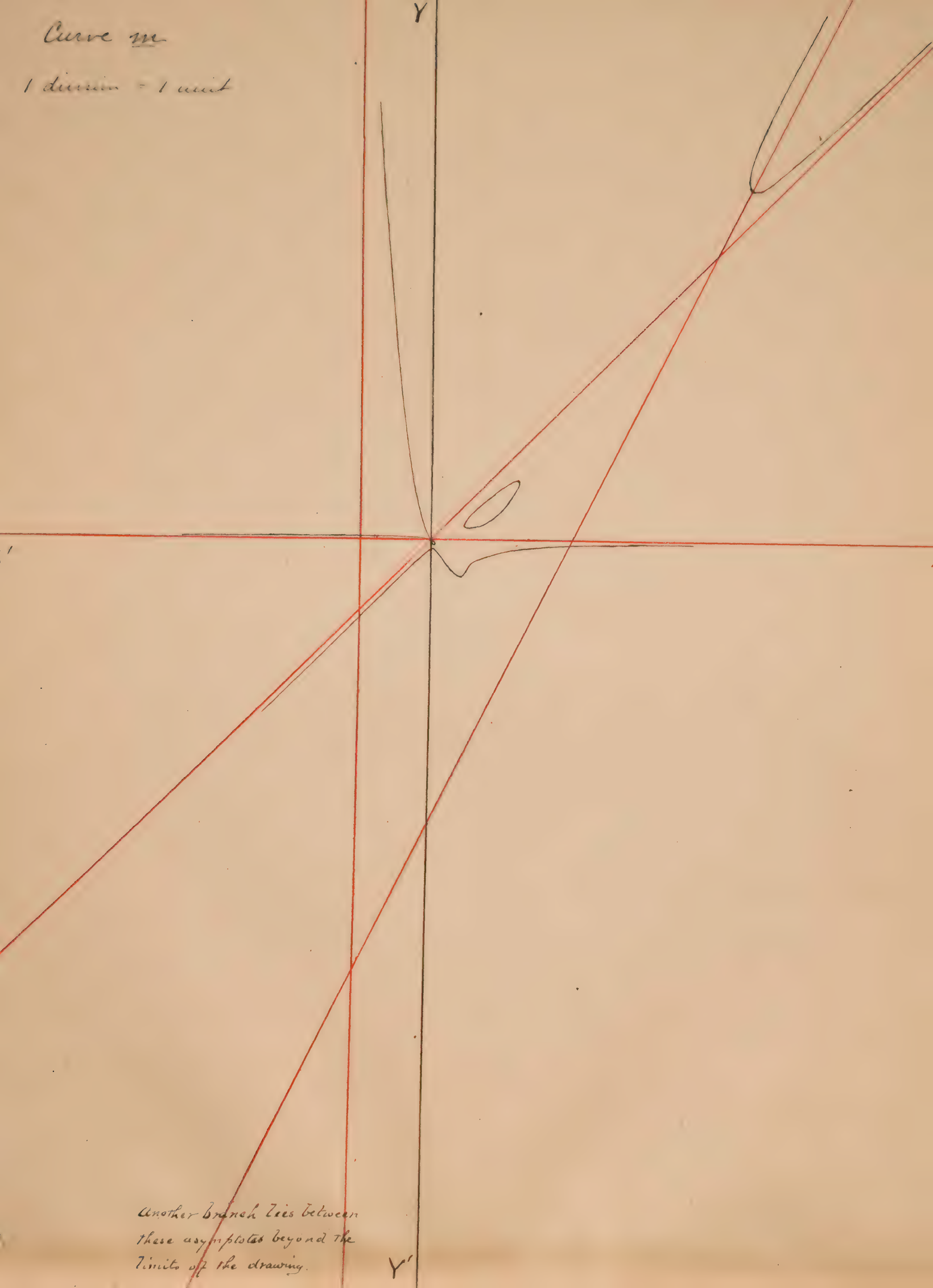
1 division = 2 unités

X'

Y'

Curve m

1 division = 1 unit



Another branch lies between these asymptotes beyond the limits of the drawing.

Y

Courbe n

1 division = 1 unité

Y'

Y

Courbe 0

4 divisions = 1 uni

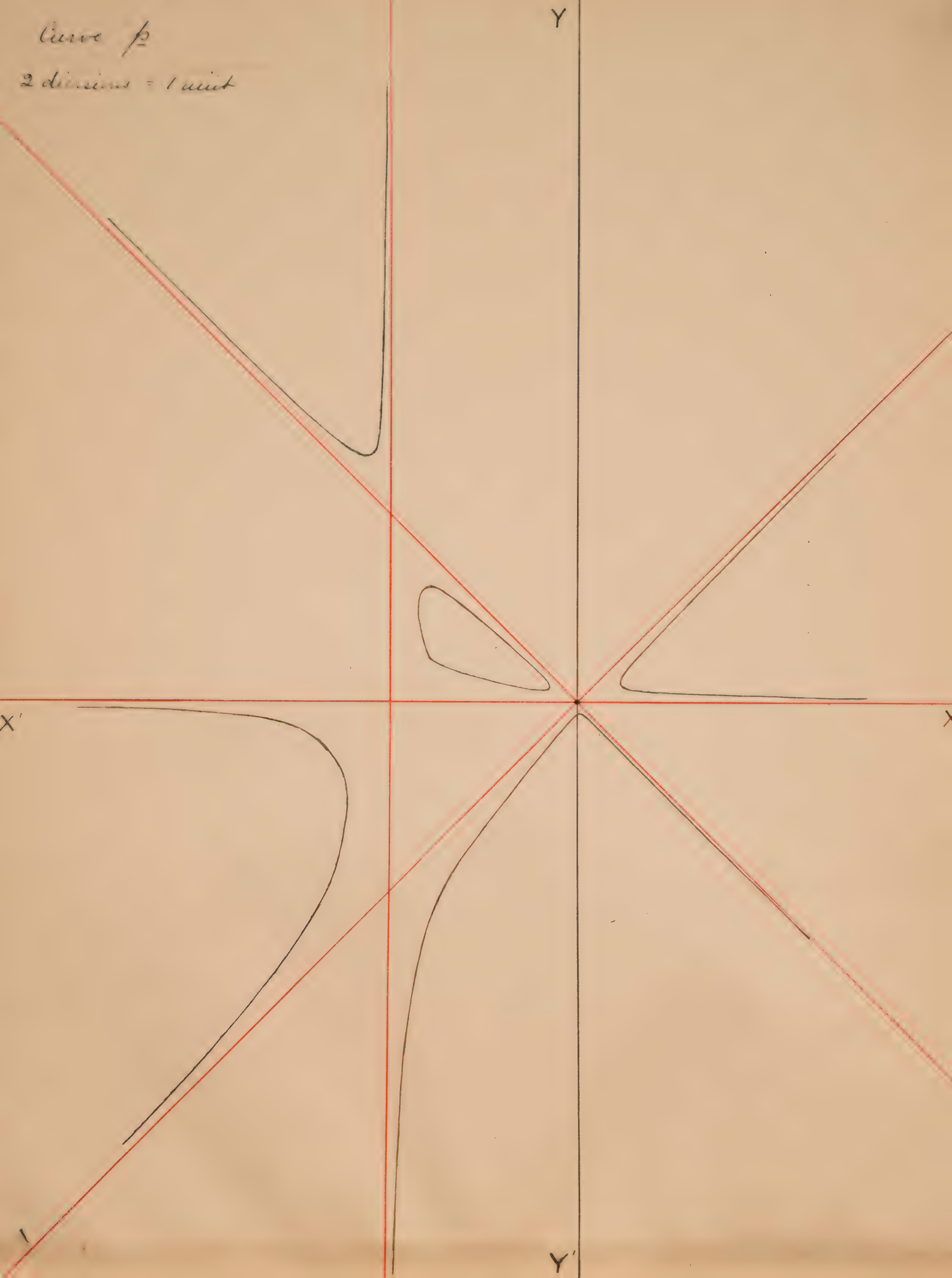
X'

X

Y'

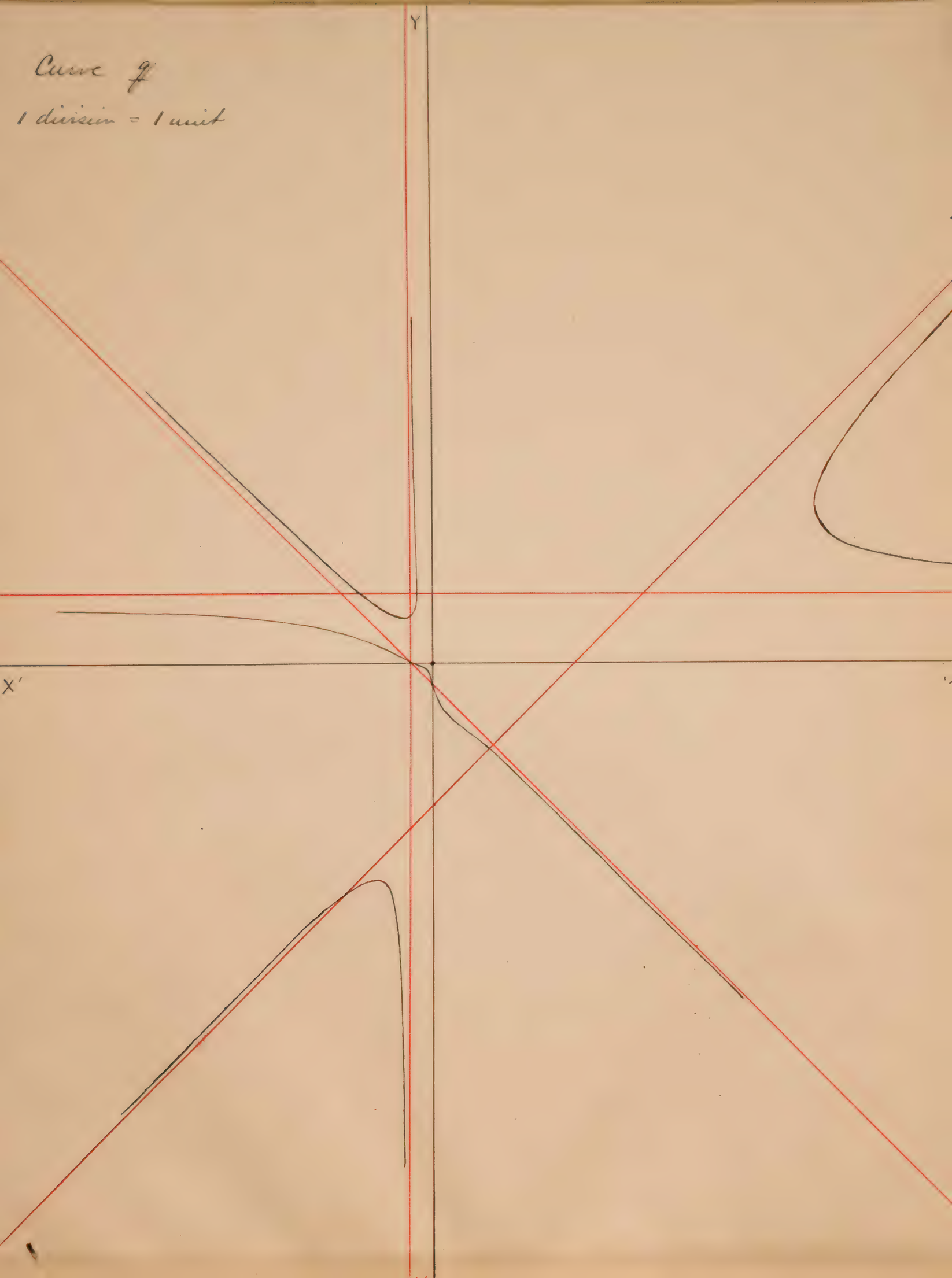
Courbe β

2 divisions = 1 unité



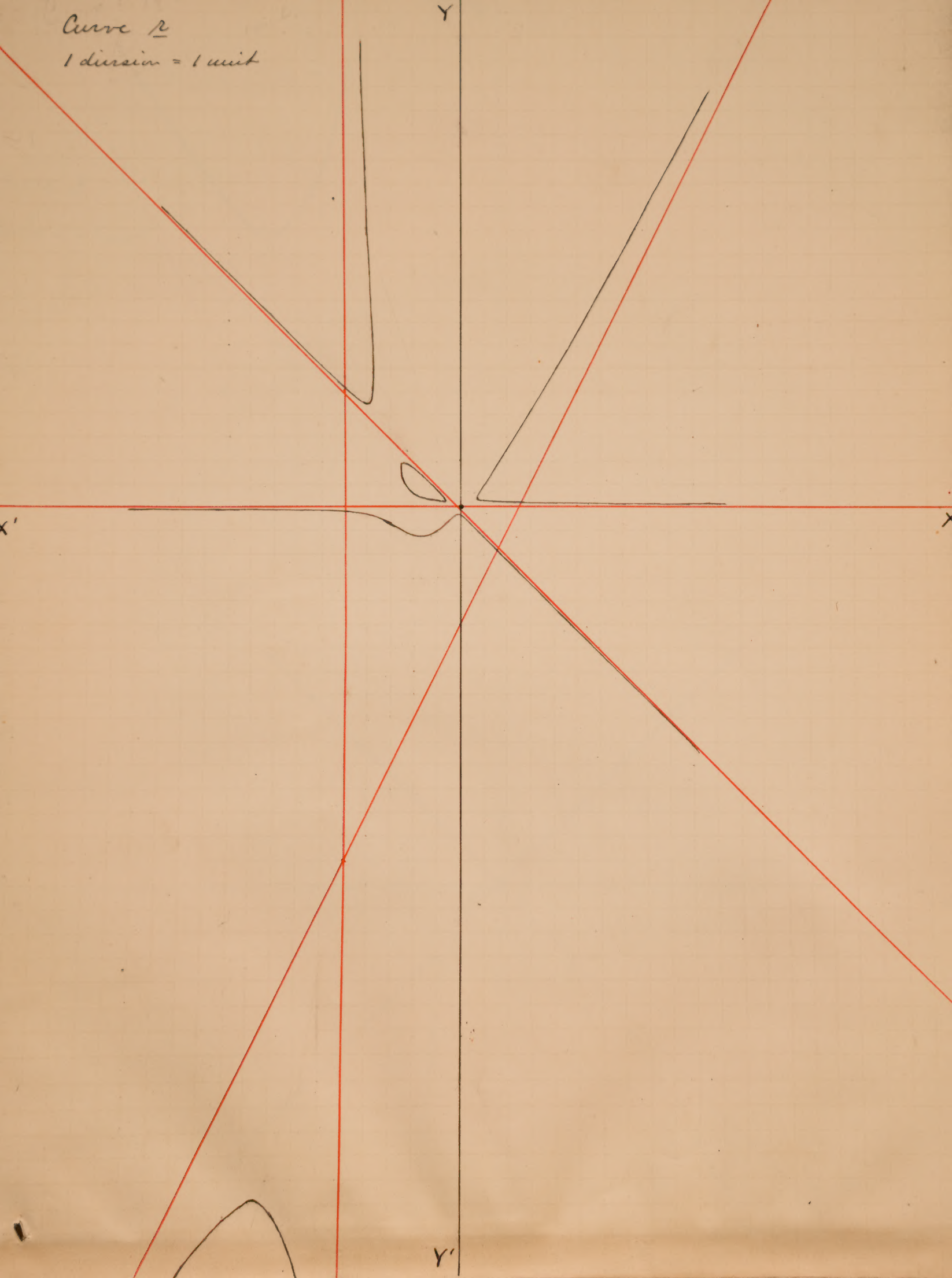
Curve 9

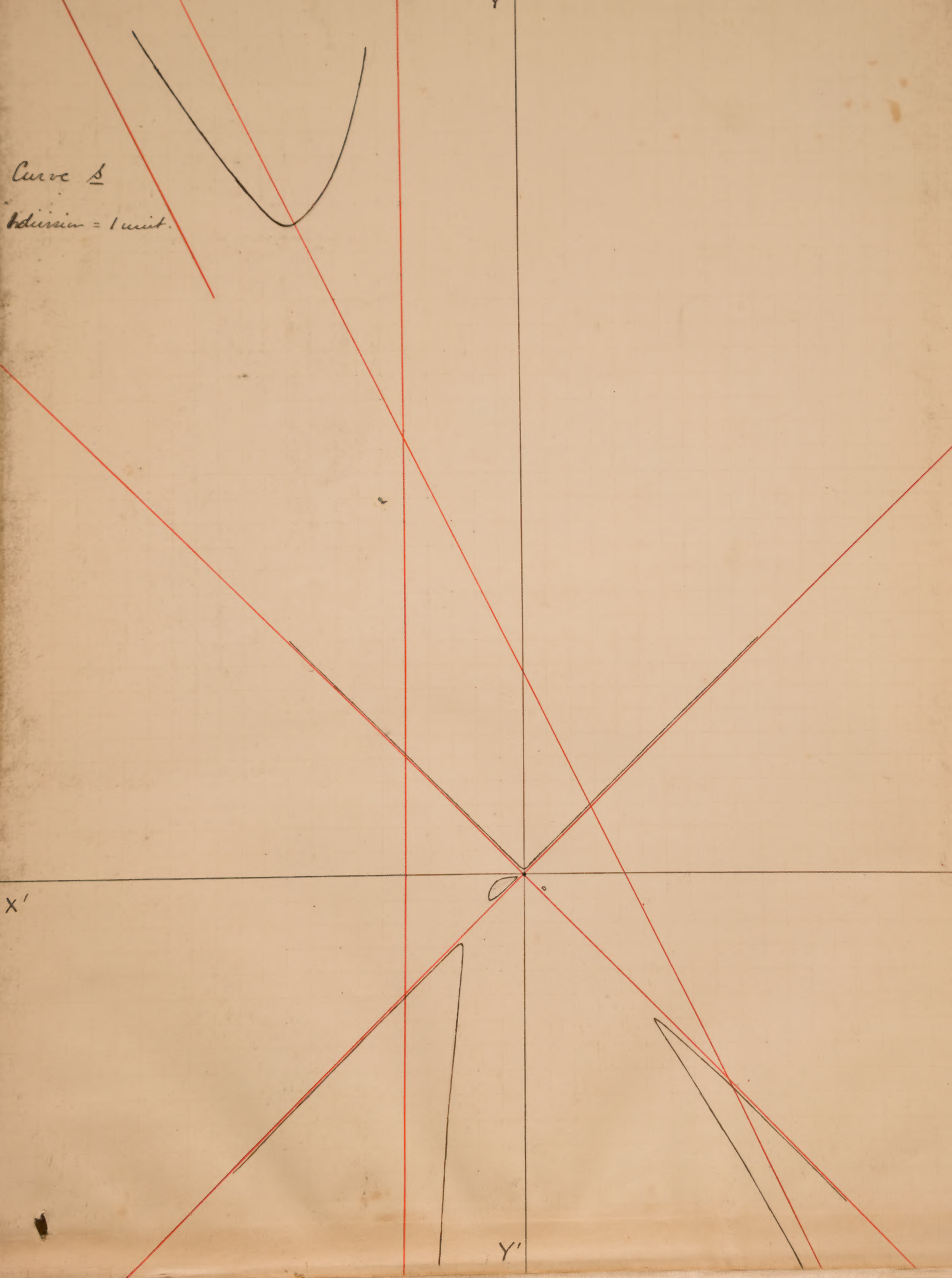
1 division = 1 unit



Curve 2

1 division = 1 unit




Curve S
Division = 1 unit.
X'
Y'

3 1198 05534 3557
N/1198/05534/3557X